PROBLEM OF EXTENSION OF AN ELASTIC SPACE CONTAINING A PLANE ANNULAR SLIT

PMM Vol. 32, No. 3, 1968, pp. 458-462

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(*Received December 28, 1967*)

The axisymmetric problem of tension of an elastic space weakened by a plane annular slit is examined. In the solution the effective asymptotic method analogous to the one developed in paper [1] is used.

Papers [2 and 3] were devoted to the development of approximate methods for the solution of the presented problem. An asymptotic method of solution of the problem of an annular slit which represents a further development of the method presented in paper [1], is applied for the solution below. The method permits to obtain the solution of the problem under examination in the form of simple equations for large and small values of parameter λ. Over some intermediate interval of variation of λ these asymptotic formulas give practically identical results, thus assuring a complete solution of the problem.

1. Solution of the problem for large λ. In the elastic space let a plane annular slit (cut) be present occupying the region: $a \leq r \leq b,\ 0 \leq \theta < 2\pi,\ z = 0$. The space is extended by forces distributed evenly at infinity, of intensity q, in the direction perpendicular to the plane of the slit. It is required to determine the form of the surface of the slit $\gamma(r)$ and the coefficient of intensity of normal stresses N, calculated without taking into account forces of cohesion at points $r = a$ and $r = b$ ($z = 0$). The problem under investigation is reduced to an auxiliary problem of an annular slit in a space, to the surface of which a normal load $\sigma_z = -q =$ const is applied, while the stresses at infinity are equal to zero.

Expressions determining $\gamma(r)$, N_a and N_b are the same for the initial and the auxiliary problems, therefore in the following we will examine the auxiliary problem. By means of the Hankel transformation the latter problem can be reduced to finding the function $\gamma(r)$ from the following integral Eq.

$$\int_a^b \rho\gamma(\rho)\,d\rho \int_0^\infty \xi^2 J_0(\xi r)\,J_0(\xi\rho)\,d\xi = -\frac{q}{\Delta} \qquad (a \leqslant r \leqslant b) \qquad \left(\Delta = \frac{E}{2(1-\nu^2)}\right) \tag{1.1}$$

Here E is Young's modulus, ν is Poisson's ratio, $J_0(x)$ is the Bessel function of zero order. Integrating both parts of Eq. (1.1) twice with respect to r, we obtain

$$\int_a^b \frac{\rho\gamma(\rho)}{r+\rho} K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right) d\rho = -\frac{\pi q}{2\Delta}\left(\frac{r^2}{4} + A\ln\frac{r}{a} + B\right) \tag{1.2}$$

Here $K(k)$ is the complete elliptic integral of the first kind, A and B are constants of integration. After making in Eq. (1.2) a substitution of variables according to Eqs. [1]

$$r = a\exp\frac{1+x}{\lambda}, \qquad \rho = a\exp\frac{1+\xi}{\lambda} \tag{1.3}$$

we obtain an integral equation of the first kind with an even difference kernel which depends on the dimensionless parameter λ

$$\int_{-1}^{1} \varphi(\xi)\, M\left(\frac{x-\xi}{\lambda}\right) d\xi = \pi f(x), \quad |x| \leqslant 1 \quad \left(\lambda = \frac{2}{\ln(b/a)}\right) \tag{1.4}$$

$$\varphi(\xi) = \rho^{3/2}\gamma(\rho), \qquad M(t) = \operatorname{sch} 0.5tK(\operatorname{sch} 0.5t) \qquad \left(t = \frac{x-\xi}{\lambda}\right)$$

$$f(x) = -\frac{q\lambda\sqrt{r}}{\Delta}\left(\frac{r^2}{4} + A\ln\frac{r}{a} + B\right) \tag{1.5}$$

Following paper [1] we represent the kernel $M(t)$ of the integral Eq. (1.4) in the following form:

$$M(t) = -\ln|t| + \sum_{i=0}^{\infty} c_i t^{2i} + \ln|t| \sum_{i=1}^{\infty} d_i t^{2i} \tag{1.6}$$

($c_0 = 2.079$, $c_1 = -0.1091$, $c_2 = 0.05352$, $d_1 = 0.0625$, $d_2 = -0.00358$ etc.)

The series in (1.6) converge for all $0 \leq t < \pi$. Substituting kernel $M(t)$ in the form (1.6) into Eq. (1.4) we obtain

$$\int_{-1}^{1} \varphi(\xi)\left[-\ln\frac{|x-\xi|}{\lambda} + c_0\right] d\xi =$$

$$= \pi\left\{f(x) - \frac{1}{\pi}\sum_{i=1}^{\infty}\frac{1}{\lambda^{2i}}\int_{-1}^{1}\varphi(\xi)\left[c_i + d_i \ln\frac{|x-\xi|}{\lambda}\right](x-\xi)^{2i}\, d\xi\right\} = \pi\psi(x) \tag{1.7}$$

Applying the transformation formula to Eq. (1.7), we obtain the integral equation of Fredholm of the second kind with respect to function $\phi(x)$:

$$\varphi(x) = -\frac{1}{\pi}\sqrt{1-x^2}\left\{\int_{-1}^{1}\frac{f'(\xi)\, d\xi}{\sqrt{1-\xi^2}(\xi-x)} - \right. \tag{1.8}$$

$$\left. -\frac{1}{\pi}\sum_{i=1}^{\infty}\frac{1}{\lambda^{2i}}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}(\xi-x)}\int_{-1}^{1}\varphi(\tau)\left[2ic_i + d_i + 2i\, d_i \ln\frac{|\xi-\tau|}{\lambda}\right](\xi-\tau)^{2i-1}\, d\tau\right\}$$

In this connection the following conditions must be fulfilled [4]:

$$\int_{-1}^{1}\frac{x\psi'(x)\, dx}{\sqrt{1-x^2}} + \int_{-1}^{1}\varphi(x)\, dx = 0, \quad \int_{-1}^{1}\frac{\psi'(x)\, dx}{\sqrt{1-x^2}} = 0 \tag{1.9}$$

$$(\ln 2\lambda + c_0)\int_{-1}^{1}\varphi(x)\, dx - \int_{-1}^{1}\frac{\psi(x)\, dx}{\sqrt{1-x^2}} = 0 \tag{1.10}$$

W may show that the first relationship (1.9) is an identity. The second relationship (1.9) and the expression (1.10) serve to determine constants A and B.

We shall seek the solution of Eq. (1.8) in the following form:

$$\varphi(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{-2n}\ln^m\lambda\,\varphi_{nm}(x) \tag{1.11}$$

Let us substitute $\phi(x)$ in the form (1.11) into the left and right parts of Eq. (1.8). Then equating expressions for equal powers of λ^{-2} and $\ln\lambda$, we obtain an infinite system of integral equations with respect to $\phi_{nm}(x)$:

$$\varphi_{00}(x) = -\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{1}\frac{f'(\xi)\, d\xi}{\sqrt{1-\xi^2}(\xi-x)} \tag{1.12}$$

$$\varphi_{10}(x) = \frac{1}{\pi^2}\sqrt{1-x^2}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}(\xi-x)}\int_{-1}^{1}\varphi_{00}(\tau)(2c_1+d_1+2d_1\ln|\xi-\tau|)(\xi-\tau)\,d\tau$$

$$\varphi_{11}(x) = -\frac{2d_1}{\pi^2}\sqrt{1-x^2}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}(\xi-x)}\int_{-1}^{1}\varphi_{00}(\tau)(\xi-\tau)\,d\tau, \quad \text{etc.}$$

Omitting intermediate calculations with respect to Eqs. (1.3) and (1.9) to (1.12), we present the final expression for determination of function $\gamma(r)$:

$$\gamma(r) = \frac{q\,(ab)^{3/4}}{\Delta r\sqrt{r}}\left(\ln\frac{r}{a}\ln\frac{b}{r}\right)^{1/2}\Phi\left(\lambda\ln\frac{r}{\sqrt{ab}}\right) \tag{1.13}$$

In the expression (1.13) the function $\Phi(t)$ is equal to (1.14)

$$\Phi(t) = 1 + (0.246+\chi)\lambda^{-2} + (0.0708+0.276\chi+\chi^2)\lambda^{-4} + [1.750+(0.336+\chi)\lambda^{-2} + (0.00922-0.176\chi)\lambda^{-4}]\,t/\lambda + [1.604+(0.264+0.385\chi)\lambda^{-2}]\,(t/\lambda)^2 + (1.029+0.119\lambda^{-2})(t/\lambda)^3 + 0.517\,(t/\lambda)^4 + 0.215\,(t/\lambda)^5 + O(\lambda^{-6}) \qquad (\chi = 0.0625\ln\lambda)$$

The coefficient of intensity of normal stresses at points $r=a$ and $r=b$ of the slit, respectively, are determined from conditions

$$N_a = \lim_{r\to a}\sqrt{a-r}_{(r<a)}\,\sigma_z = \lim_{r\to a}\sqrt{r-a}_{(r>a)}\,\Delta\frac{d\gamma}{dr}$$
$$N_b = \lim_{r\to b}\sqrt{r-b}_{(r>b)}\,\sigma_z = -\lim_{r\to b}\sqrt{b-r}_{(r<b)}\,\Delta\frac{d\gamma}{dr} \tag{1.15}$$

Substituting $\gamma(r)$ in the form (1.13) into condition (1.15) we obtain

$$N_a = \frac{q\sqrt{a}\,\Phi(-1)}{\sqrt{2\lambda}\exp(-2.5/\lambda)}, \qquad N_b = \frac{q\sqrt{b}\,\Phi(1)}{\sqrt{2\lambda}\exp(2.5/\lambda)} \tag{1.16}$$

2. Solution of the problem for small λ (method of successive approximations). Let us differentiate with respect to r both parts of Eq. (1.2). Then integrating by parts and making use of the condition

$$\gamma(a) = \gamma(b) = 0 \tag{2.1}$$

we obtain

$$\int_a^b \rho\gamma'(\rho)\,d\rho\int_0^\infty J_1(\xi r)J_1(\xi\rho)\,d\xi = -\frac{q}{\Delta}\left(\frac{1}{2}r + A\frac{1}{r}\right) \qquad (a\leqslant r\leqslant b) \tag{2.2}$$

The integral Eq. (2.2) is equivalent to the following system of two integral Eqs.:

$$\int_0^b \rho\gamma_1'(\rho)\,d\rho\int_0^\infty J_1(\xi r)J_1(\xi\rho)\,d\xi = -\frac{qr}{2\Delta} + \int_b^\infty \rho\gamma_2'(\rho)\,d\rho\int_0^\infty J_1(\xi r)J_1(\xi\rho)\,d\xi \qquad (0\leqslant r\leqslant b)$$
$$\int_a^\infty \rho\gamma_2'(\rho)\,d\rho\int_0^\infty J_1(\xi r)J_1(\xi\rho)\,d\xi = -\frac{Aq}{\Delta r} + \int_0^a \rho\gamma_1'(\rho)\,d\rho\int_0^\infty J_1(\xi r)J_1(\xi\rho)\,d\xi \qquad (a\leqslant r<\infty) \tag{2.3}$$

under the condition that

$$\gamma'(r) = \gamma_1'(r) + \gamma_2'(r) \tag{2.4}$$

Let us introduce functions $\Gamma_1(\xi)$ and $\Gamma_2(\xi)$ in the following manner:

$$\int_0^\infty \xi\Gamma_1(\xi)J_1(\xi r)\,d\xi = \begin{cases}\gamma_1'(r) & (0<r<b)\\ -\gamma_2'(r) & (b<r<\infty)\end{cases} \tag{2.5}$$

$$\int_0^\infty \xi\Gamma_2(\xi)J_1(\xi r)\,d\xi = \begin{cases}-\gamma_1'(r) & (0<r<a)\\ \gamma_2'(r) & (a<r<\infty)\end{cases} \tag{2.6}$$

In this case the system (2.3) is reduced to two dual integral equations with respect to functions $\Gamma_1(\xi)$ and $\Gamma_2(\xi)$:

$$\int_0^\infty \Gamma_1(\xi) J_1(\xi r)\, d\xi = -\frac{qr}{2\Delta}, \quad (0<r<b) \qquad \int_0^\infty \xi\Gamma_1(\xi) J_1(\xi r)\, d\xi = -\gamma_2'(r) \quad (b<r<\infty) \tag{2.7}$$

$$\int_0^\infty \xi\Gamma_2(\xi) J_1(\xi r)\, d\xi = -\gamma_1'(r), \quad (0<r<a) \qquad \int_0^\infty \Gamma_2(\xi) J_1(\xi r)\, d\xi = -\frac{qA}{\Delta r} \quad (a<r<\infty) \tag{2.8}$$

Applying the transformation formula to Eqs. (2.7) and (2.8), and substituting the obtained functions $\Gamma_1(\xi)$ and $\Gamma_2(\xi)$ into (2.5) and (2.6), we obtain a system of two integral equations of the second kind with respect to functions $\gamma_1'(r)$ and $\gamma_2'(r)$:

$$\gamma_1'(r) = -\frac{2q}{\pi\Delta}\frac{r}{\sqrt{b^2-r^2}} + \frac{2}{\pi}\frac{r}{\sqrt{b^2-r^2}}\int_b^\infty \frac{\gamma_2'(\rho)\sqrt{\rho^2-b^2}}{\rho^2-r^2}d\rho \quad (0\leqslant r\leqslant b)$$

$$\gamma_2'(r) = \frac{2qabC}{\pi\Delta r\sqrt{r^2-a^2}} + \frac{2}{\pi}\frac{r}{\sqrt{r^2-a^2}}\int_0^a \frac{\gamma_1'(\rho)\sqrt{a^2-\rho^2}}{r^2-\rho^2}d\rho \quad (a\leqslant r<\infty) \tag{2.9}$$

It is evident from (2.9) that the presented method is applicable to small values of λ, because small λ correspond to wide annuli, i.e. to the case when either a is small or b is large for $b \gg a$. We shall seek the solution of system (2.9) in the form

$$\gamma_1'(r) = \sum_{i=0}^\infty \gamma_{1i}'(r), \qquad \gamma_2'(r) = \sum_{i=0}^\infty \gamma_{2i}'(r) \tag{2.10}$$

$$\gamma_{10}'(r) = -\frac{2q}{\pi\Delta}\frac{r}{\sqrt{b^2-r^2}}, \qquad \gamma_{1,\,i+1}'(r) = \frac{2}{\pi}\frac{r}{\sqrt{b^2-r^2}}\int_b^\infty \frac{\gamma_{2i}'(\rho)\sqrt{\rho^2-b^2}}{\rho^2-r^2}d\rho$$

$$\gamma_{20}'(r) = \frac{2qabC}{\pi\Delta r\sqrt{r^2-a^2}}, \qquad \gamma_{2,\,i+1}'(r) = \frac{2}{\pi}\frac{r}{\sqrt{r^2-a^2}}\int_0^a \frac{\gamma_{1i}'(\rho)\sqrt{a^2-\rho^2}}{r^2-\rho^2}d\rho$$

$$(i=0,1,\ldots)$$

Limiting ourselves to calculation of functions $\gamma_{11}'(r)$ and $\gamma_{21}'(r)$ and taking into account of (2.10) and (2.4), we obtain

$$\gamma'(r) = \frac{2q}{\pi\Delta}\frac{r}{\sqrt{b^2-r^2}}\left[\frac{1}{\pi}\arccos\frac{r^2(a^2+b^2)-2a^2b^2}{r^2(b^2-a^2)} + \frac{b^2C}{\pi r^2}\ln\frac{b+a}{b-a} - 1\right] -$$
$$-\frac{2qabC}{\pi\Delta r\sqrt{r^2-a^2}}\left[\frac{1}{\pi}\arccos\frac{a^2+b^2-2r^2}{b^2-a^2} + \frac{r^2}{\pi abC}\ln\frac{b+a}{b-a} - 1\right] \quad (a\leqslant r\leqslant b) \tag{2.11}$$

Substituting $\gamma'(r)$ in the form (2.11) into condition (1.15) we obtain

$$N_a = \frac{q\sqrt{2b}}{\pi}\left(\frac{C}{\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}}{\pi}\ln\frac{1+\varepsilon}{1-\varepsilon}\right), \qquad N_b = \frac{q\sqrt{2b}}{\pi}\left(1 - \frac{C}{\pi}\ln\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{2.12}$$

$$(\varepsilon = \exp(-2/\lambda))$$

Integrating function $\gamma'(r)$ in the form (2.11) we obtain

$$\gamma(r) = \frac{2q}{\pi\Delta}\sqrt{b^2-r^2} + \frac{4q}{\pi^2\Delta}C\left\{b\arcsin\frac{a}{r}\arcsin\frac{r}{b} -\right.$$
$$-\left[\left(\frac{r}{b} + \frac{2}{3}\frac{r^3}{b^3} + \frac{8}{15}\frac{r^5}{b^5}\right)\arcsin\frac{a}{r} - \varepsilon - \frac{2}{3}\frac{\varepsilon r^2}{b^2} - \frac{1}{9}\varepsilon^3 - \frac{8}{15}\frac{\varepsilon r^4}{b^4} - \frac{4}{45}\frac{\varepsilon^3 r^2}{b^2} -\right.$$

$$-\frac{1}{25}\varepsilon^5 + O(\varepsilon^7)\Big]\sqrt{b^2-r^2}\Big\} + \left(C - \frac{2}{\pi}\right)\frac{2qb}{\pi\Delta}\arc\cos\frac{a}{r} -$$

$$-\frac{2q}{\pi^2\Delta}\left[\sqrt{r^2-a^2}\ln\frac{1+\varepsilon}{1-\varepsilon} + \sqrt{b^2-r^2}\arccos\frac{r^2(\varepsilon^2+1)-2a^2}{r^2(1-\varepsilon^2)}\right] + D\frac{qb}{\Delta} \qquad (2.13)$$

The constants C and D are determined from conditions (2.1):

$$C = \frac{1}{\pi}\,\frac{2\arccos\varepsilon + \sqrt{1-\varepsilon^2}\ln[(1+\varepsilon)(1-\varepsilon)^{-1}]}{\arccos\varepsilon + (0.3634\varepsilon + 0.1715\varepsilon^3 + 0.1117\varepsilon^5 + O(\varepsilon^7))\sqrt{1-\varepsilon^2}}$$
$$D = 2\pi^{-2}\{2\arccos\varepsilon + \sqrt{1-\varepsilon^2}\ln[(1+\varepsilon)(1-\varepsilon)^{-1}]\} - C \qquad (2.14)$$

3. Solution of the problem for small λ (method of products). Numerical analysis of the problem. A solution of Eq. (1.1) which is applicable to small values of parameter λ can also be obtained in the form of combination of solutions of integral Eqs. [1]

$$\int_0^b \rho\gamma_3(\rho)\,d\rho\int_0^\infty \xi^2 J_0(\xi r)J_0(\xi\rho)\,d\xi = \frac{q}{\Delta} \qquad (0\leqslant r\leqslant b) \qquad (3.1)$$

$$\int_a^\infty \rho\gamma_4(\rho)\,d\rho\int_0^\infty \xi^2 J_0(\xi r)J_0(\xi\rho)\,d\xi = \frac{q}{\Delta}J_0(\beta r) \qquad (a\leqslant r<\infty) \qquad (3.2)$$

in the following form:

$$\gamma(r) = \lim_{\beta\to 0}\frac{\gamma_3(r)\gamma_4(r)}{\gamma_5(r)} \qquad (3.3)$$

Here $\gamma_5(r)$ is a degenerate solution of Eq. (3.2) which represents the first term of the asymptote $\gamma_4(r)$ for $r/a\to\infty$.

The solution of Eqs. (3.1) and (3.2) can be obtained if these equations are reduced to their equivalent dual integral equations, for example as this was done in the solution of Eqs. (2.3). We present the final expressions

$$\gamma_3(r) = \frac{2q}{\pi\Delta}\sqrt{b^2-r^2}, \qquad \gamma_4(r) = \frac{2q}{\pi\Delta\beta}\left(\arccos\frac{a}{r} + O(\beta^2)\right) \qquad (3.4)$$

It follows from the second relationship (3.4) that

$$\gamma_5(r) = q(\Delta\beta)^{-1}[1 + O(\beta^2)] \qquad (3.5)$$

Substituting $\gamma_3(r)$, $\gamma_4(r)$ and $\gamma_5(r)$ in (3.3) we obtain

$$\gamma(r) = \Delta^{-1}4q\pi^{-2}\sqrt{b^2-r^2}\arccos a/r \qquad (3.6)$$

The expression which determines the coefficient of intensity of normal stresses at points $r=a$ and $r=b$ are obtained from relationships (3.6) and (1.15), respectively

$$N_a = 2\sqrt{2b}q\pi^{-2}\varepsilon^{-1/2}\sqrt{1-\varepsilon^2}, \qquad N_b = 2\sqrt{2b}q\pi^{-2}\arccos\varepsilon \qquad (3.7)$$

Completed calculations showed that Eqs. (1.13) and (1.16) can be used reliably for $2\leq\lambda<\infty$, Eqs. (2.12) and (2.13) for $0<\lambda\leq 2$ and Eqs. (3.6) and (3.7) for $0<\lambda<0.75$. A numerical analysis of equations for N_a and N_b shows that N_a is always larger than N_b. It follows from this that the form of the annular slit is unstable. The development of the annular gap for monotonous increase of the load q applied at infinity starts at points of the inner contour and the annular gap transforms into a circular gap of radius $r=b$.

We present the values of quantities $\gamma^* = (qb)^{-1}\Delta\gamma(0.5(a+b))$, $N_a^* = (q\sqrt{b})^{-1}N_a$ and $N_b^* = (q\sqrt{b})^{-1}N_b$, calculated for $\lambda = 2$ (the first two columns) and $\lambda = 0.75$ (the third and fourth column) from equations of Sections 1, 2 and 3:

Section	1	2	2	3
$\gamma^* =$	0.326	0.323	0.503	0.493
$N_a^* =$	0.493	0.486	1.113	1.084
$N_b^* =$	0.372	0.369	0.437	0.430

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Translated by B.D.